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SOLUTION OF THE BOUNDARY-VALUE PROBLEM FOR THE EQUATION OF RADIATION TRANSFER IN A LIMITED VOLUME OF A DISPERSION MEDIUM WITH A LINEAR SCATTERING INDICATRIX

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We propose a method for solving the boundary-value problem for the equation of radiation transfer with a linear indicatrix, with the boundary conditions being exact in the formulation of this problem. The results of the numerical experiments based on the solution of the radiation-transfer problem are presented.

Introduction. Many problems of the theory of radiation transfer in dispersion media are solved on the assumption that the medium has an infinite extent or occupies an infinite half-space [1, 2]. However, real nature objects and technical dispersion media have, as a rule, a limited extent. Nevertheless, the mechanisms of radiation transfer in media of limited dimensions are still not clearly understood. Therefore, investigations for three-dimensional volumes present a topical problem of the optics of dispersion media, the physics of protection of reactors, astrophysics, atmospheric optics, etc. [3–5]. In the present paper, we solve the boundary-value problem for the radiation-transfer equation (RTE) with a linear scattering indicatrix. We find the solution of the RTE in the form

$$I(x, y, z, \Omega_1, \Omega_2, \Omega_3) = J_0(x, y, z) + J_1(x, y, z)\,\Omega_1 + J_2(x, y, z)\,\Omega_2 + J_3(x, y, z)\,\Omega_3\,, \tag{1}$$

where $\mathbf{\Omega} = (\Omega_1(\vartheta, \varphi), \Omega_2(\vartheta, \varphi), \Omega_3(\vartheta, \varphi))$ is the unit vector of direction. The functions $J_0(\mathbf{r})$, $J_1(\mathbf{r})$, $J_2(\mathbf{r})$, and $J_3(\mathbf{r})$ depend only on the spatial variables $\mathbf{r} = (x, y, z)$ and are found by solving the auxiliary boundary-value problem for the modified Helmholz equation. Having solved the boundary-value problem for the RTE, where the medium volume is chosen in the form of a parallelepiped, we will obtain the results of numerical experiments.

Formulation of the Boundary-Value Problem for the Radiation-Transfer Equation. Let a monochromatic, monodirectional light beam be normally incident on a spatially three-dimensional volume of a dispersion medium with a linear scattering indicatrix. We chose the medium volume in the form of a rectangular parallelepiped oriented along the axes of the Cartesian coordinate system xyz. The surface Γ of the medium is defined by six faces given by the equations: $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, $z = c$. To each face there correspond six internal normals having the following coordinates: $n_1 = 1$, $n_2 = 0$, $n_3 = 0$; $n_1 = -1$, $n_2 = 0$, $n_3 = 0$; $n_1 = 0$, $n_2 = 1$, $n_3 = 0$; $n_1 = 0$, $n_2 = -1$, $n_3 = 0$; $n_1 = 0$, $n_2 = 0$, $n_3 = 1$; $n_1 = 0$, $n_2 = 0$, $n_3 = -1$. The volume of the medium $V = abc$.

The integrodifferential transfer equation [1]

$$(\mathbf{\Omega}, \nabla)\, I(\mathbf{r}, \mathbf{\Omega}) + \varepsilon I(\mathbf{r}, \mathbf{\Omega}) = \frac{\varepsilon\Lambda}{4\pi} \int_{\Omega'} \kappa(\mathbf{\Omega}, \mathbf{\Omega}')\, I(\mathbf{r}, \mathbf{\Omega}')\, d\Omega' + B_1(\mathbf{r}, \mathbf{\Omega})\,, \tag{2}$$

complemented by the boundary condition

$$I(\mathbf{r}_\Gamma, \mathbf{\Omega}) = 0 \quad \text{at} \quad (\mathbf{n}_\Gamma, \mathbf{\Omega}) > 0\,, \tag{3}$$

defines the boundary-value problem of the theory of radiation transfer in a dispersion medium under multiple scattering. In Eq. (2) $I(\mathbf{r}, \mathbf{\Omega})$ is the diffuse radiation intensity as a function of the spatial coordinates $\mathbf{r} = (x, y, z)$ and the

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direction $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ ($\Omega_1(\vartheta, \varphi) = \sin\vartheta \cos\varphi$, $\Omega_2(\vartheta, \varphi) = \sin\vartheta \sin\varphi$, $\Omega_3(\vartheta, \varphi) = \cos\vartheta$, ϑ and φ being the polar and azimuth angles of the spherical coordinate system). The scalar product is written as

$$(\mathbf{\Omega}, \mathbf{\Omega}') = \Omega_1\Omega_1' + \Omega_2\Omega_2' + \Omega_3\Omega_3' = \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\varphi - \varphi') = \cos\theta\ , \tag{4}$$

where θ is the scattering angle, i.e., the angle between vectors Ω and Ω'. Let us denote the scattering indicatrix as $\kappa(\mu)$ $(-1 \le \mu \le 1)$; here

$$\int_{-1}^{1} \kappa(\mu)\, d\mu = 2\ . \tag{5}$$

In our case, $\kappa(\mu) = 1 + g\mu$ (g is the given coefficient) [2]:

$$B_1(r, \Omega) = \frac{\varepsilon\Lambda}{4\pi} F\pi \exp(-\varepsilon z)\, x(\cos\vartheta) = \frac{\varepsilon\Lambda F}{4} \exp(-\varepsilon z)\,(1 + g\cos\vartheta) \tag{6}$$

is the function of the single scattering sources.

Form of the RTE Solution for the Linear Scattering Indicatrix. We find the solution of the radiation-transfer equation in the form

$$I = J_0 + J_1\Omega_1 + J_2\Omega_2 + J_3\Omega_3\ . \tag{7}$$

Proceeding from the problem symmetry (a constant radiation flux is incident on the parallelepiped), we assume that $\frac{\partial I}{\partial x} \approx 0$ and $\frac{\partial I}{\partial y} \approx 0$. Choosing the direction vector $\Omega_1 = p$, $\Omega_2 = q$, $\Omega_3 = 0$, $p^2 + q^2 = 1$, rewrite the radiation-transfer equation as

$$I(\mathbf{r}, \mathbf{\Omega}) = \frac{\Lambda}{4\pi} \int_{\Omega'} \kappa(\mathbf{\Omega}, \mathbf{\Omega}')\, I(\mathbf{r}, \mathbf{\Omega}')\, d\Omega' + \frac{\Lambda F}{4} \exp(-\varepsilon z)\,(1 + g\Omega_3)\ . \tag{8}$$

Hence, substituting into the integral the function $\kappa(\mu) = 1 + g\mu$, we obtain relation (7). We will seek the RTE solution in the form of (7), and to determine the functions J_0, J_1, J_2, and J_3, we derive the equation in partial derivatives for J_0.

Method for Solving the Boundary-Value Problem for the RTE with a Linear Scattering Indicatrix. Calculate the integral

$$\frac{1}{4\pi} \int_{\Omega'} (1 + g(\Omega_1\Omega_1' + \Omega_2\Omega_2' + \Omega_3\Omega_3'))\,(J_0 + J_1\Omega_1' + J_2\Omega_2' + J_3\Omega_3')\, d\Omega' = J_0 + \frac{g}{3}(J_1\Omega_1 + J_2\Omega_2 + J_3\Omega_3)\ . \tag{9}$$

Integration with respect to the solid angle yields, in the case of the RTE, the relation

$$\frac{4\pi}{3}\left\{\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z}\right\} + 4\pi\varepsilon J_0 = \varepsilon\Lambda J_0 + \varepsilon\Lambda\pi F \exp(-\varepsilon z)\ , \tag{10}$$

and for expression (2) multiplied by Ω_1, Ω_2, Ω_3 — the equation

$$\frac{4\pi}{3}\frac{\partial J_0}{\partial x} + \frac{4\pi}{3}\varepsilon J_1 = \frac{g}{3} J_1 \frac{4\pi}{3}\varepsilon\Lambda\ , \tag{11}$$

$$\frac{4\pi}{3}\frac{\partial J_0}{\partial y} + \frac{4\pi}{3}\varepsilon J_2 = \frac{g}{3} J_2 \frac{4\pi}{3}\varepsilon\Lambda\ , \tag{12}$$

$$\frac{4\pi}{3}\frac{\partial J_0}{\partial z}+\frac{4\pi}{3}\varepsilon J_3=\frac{g}{3}J_3\frac{4\pi}{3}\varepsilon\Lambda+\frac{\varepsilon\Lambda F}{4}g\frac{4\pi}{3}\exp(-\varepsilon z)\,. \tag{13}$$

From the system of equations (11)–(13) it follows that

$$J_1=\frac{1}{\varepsilon\left(\Lambda\frac{g}{3}-1\right)}\frac{\partial J_0}{\partial x}\,, \tag{14}$$

$$J_1=\frac{1}{\varepsilon\left(\Lambda\frac{g}{3}-1\right)}\frac{\partial J_0}{\partial y}\,, \tag{15}$$

$$J_3=\frac{1}{\varepsilon\left(\Lambda\frac{g}{3}-1\right)}\frac{\partial J_0}{\partial z}-\frac{1}{\varepsilon\left(\Lambda\frac{g}{3}-1\right)}\frac{\varepsilon\Lambda F}{4}g\exp(-\varepsilon z)\,. \tag{16}$$

Substituting (14)–(16) into Eq. (10), we get

$$\Delta J_0-3\varepsilon^2(\Lambda-1)\left(\Lambda\frac{g}{3}-1\right)J_0=\frac{\varepsilon^2\Lambda F}{4}\exp(-\varepsilon z)\left(3\left(\Lambda\frac{g}{3}-1\right)-g\right). \tag{17}$$

Thus, to find J_0, it is necessary to solve the modified Helmholz equation (17) with boundary conditions on the parallelepiped.

To derive the boundary conditions, the input relation is

$$I\left(\mathbf{r}_\Gamma,\boldsymbol{\Omega}\right)=0\quad\text{for}\quad\left(\mathbf{n}_\Gamma,\boldsymbol{\Omega}\right)>0\,, \tag{18}$$

where $I(\mathbf{r}_\Gamma, \boldsymbol{\Omega})$ is the diffuse radiation intensity at the parallelepiped boundary. From (18) follows the boundary condition in the form of an integral for the illuminance $E(\mathbf{r}_\Gamma, \mathbf{n}_\Gamma)$ of the boundary Γ of the parallelepiped:

$$E\left(\mathbf{r}_\Gamma,\mathbf{n}_\Gamma\right)=\iint_{(\mathbf{n}_\Gamma,\boldsymbol{\Omega})>0}I\left(\mathbf{r}_\Gamma,\boldsymbol{\Omega}\right)\left(\mathbf{n}_\Gamma,\boldsymbol{\Omega}\right)d\Omega=0\,, \tag{19}$$

from which it follows that the boundary Γ illumination by the outside radiation is absent.

Next we substitute into integral (19) the intensity $I(\mathbf{r}, \boldsymbol{\Omega})$ in the form of sum (17) and obtain

$$E\left(\mathbf{r}_\Gamma,\mathbf{n}_\Gamma\right)=J_0\left(\mathbf{r}_\Gamma\right)\iint_{(\mathbf{n}_\Gamma,\boldsymbol{\Omega})>0}\left(\mathbf{n}_\Gamma,\boldsymbol{\Omega}\right)d\Omega+J_1\left(\mathbf{r}_\Gamma\right)\iint_{(\mathbf{n}_\Gamma,\boldsymbol{\Omega})>0}\Omega_1\left(\mathbf{n}_\Gamma,\boldsymbol{\Omega}\right)d\Omega+$$

$$+J_2\left(\mathbf{r}_\Gamma\right)\iint_{(\mathbf{n}_\Gamma,\boldsymbol{\Omega})>0}\Omega_2\left(\mathbf{n}_\Gamma,\boldsymbol{\Omega}\right)d\Omega+J_3\left(\mathbf{r}_\Gamma\right)\iint_{(\mathbf{n}_\Gamma,\boldsymbol{\Omega})>0}\Omega_3\left(\mathbf{n}_\Gamma,\boldsymbol{\Omega}\right)d\Omega=$$

$$=J_0\left(\mathbf{r}_\Gamma\right)\pi+J_1\left(\mathbf{r}_\Gamma\right)\frac{2\pi}{3}n_1+J_2\left(\mathbf{r}_\Gamma\right)\frac{2\pi}{3}n_2+J_3\left(\mathbf{r}_\Gamma\right)\frac{2\pi}{3}n_3\,. \tag{20}$$

As before, another equation of the system is the relation

$$\mathbf{J}(\mathbf{r})=\frac{1}{\varepsilon\left(\Lambda\frac{g}{3}-1\right)}\nabla J_0(\mathbf{r})\,. \tag{21}$$

Substituting (21) into (20), we obtain

$$E\left(\mathbf{r}_\Gamma, \mathbf{n}_\Gamma\right) = \pi J_0\left(\mathbf{r}_\Gamma\right) - \frac{2\pi}{3}\frac{1}{\varepsilon\left(1 - \Lambda\frac{g}{3}\right)}\left(\nabla J_0\left(\mathbf{r}_\Gamma\right), \mathbf{n}_\Gamma\left(\mathbf{r}_\Gamma\right)\right) = 0 \tag{22}$$

or

$$\left(\mathbf{n}_\Gamma, \nabla\right) J_0\left(\mathbf{r}_\Gamma\right) - \frac{3}{2}\varepsilon\left(1 - \Lambda\frac{g}{3}\right) J_0\left(\mathbf{r}_\Gamma\right) = 0 \tag{23}$$

is the sought boundary condition. Thus, the solution of the radiation-transfer equation has reduced to the solution of the boundary-value problem for the modified Helmholz equation:

$$\begin{gathered} \Delta J_0 - k^2 J_0 = -K\exp(-\varepsilon z), \\ \left(\mathbf{n}_\Gamma, \nabla\right) J_0\left(\mathbf{r}_\Gamma\right) - \eta J_0\left(\mathbf{r}_\Gamma\right) = 0. \end{gathered} \tag{24}$$

Here $k^2 = 3\varepsilon^2(\Lambda - 1)\left(\Lambda\frac{g}{3} - 1\right)$, $K = \frac{\varepsilon^2 \Lambda F}{4}\left(3\left(\frac{g}{3} - 1\right) - g\right)$, and $\eta = \frac{3}{2}\varepsilon\left(1 - \Lambda\frac{g}{3}\right)$ are positive constants.

Let us solve problem (24) by the Fourier method, namely: we shall seek the function $J_0(x, y, z)$ in the form

$$J_0(x, y, z) = \sum_{i,j=1} Z_{ij}(z)\, \overline{X}_i(x)\, \overline{Y}_j(y), \tag{25}$$

where the functions $\overline{X}_i(x)\overline{Y}_j(y)$ form a complete orthonormalized system. Thus, $X_i(x)$ denotes the solutions to the following Sturm–Liouville problem [6]:

$$\begin{aligned} & X'' + \lambda_x^2 X = 0; \\ & X'(x) - \eta X(x) = 0, \quad x = 0; \\ & X'(x) + \eta X(x) = 0, \quad x = a. \end{aligned} \tag{26}$$

Thus,

$$X_i(x) = \sin\left(\frac{\mu_i}{a}x + \theta_i\right), \quad \theta_i = \arctan\frac{\mu_i}{\eta a}, \quad \lambda_{xi} = \frac{\mu_i}{a}, \tag{27}$$

here μ_i denotes the solutions of the transcendental equation

$$\cot\mu = \frac{1}{2}\left(\frac{\mu}{\eta a} - \frac{\eta a}{\mu}\right); \tag{28}$$

$$\|X_i\|^2 = \frac{a}{2} + \frac{\eta}{\lambda_{xi}^2 + \eta^2}, \quad \overline{X}_i(x) = \frac{X_i(x)}{\|X_i\|}.$$

Likewise, for the function $Y_i(y)$ we write the system

$$\begin{aligned} & Y'' + \lambda_y^2 Y = 0; \\ & Y'(y) - \eta Y(y) = 0, \quad y = 0; \\ & Y'(y) + \eta Y(y) = 0, \quad y = b. \end{aligned} \tag{29}$$

The solutions of (29) will be the functions

$$Y_j(y) = \sin\left(\frac{\mu_j}{b} y + \theta_j\right), \quad \theta_j = \arctan\frac{\mu_j}{\eta b}, \quad \lambda_{yj} = \frac{\mu_j}{b}; \tag{30}$$

μ_j denotes the solutions of the transcendental equation

$$\cot\mu = \frac{1}{2}\left(\frac{\mu}{\eta b} - \frac{\eta b}{\mu}\right); \tag{31}$$

$$\|Y_j\|^2 = \frac{b}{2} + \frac{\eta}{\lambda_{yj}^2 + \eta^2}, \quad \overline{Y}_j(y) = \frac{Y_j(y)}{\|Y_j\|}.$$

Find the functions $Z_{ij}(z)$. To this end, according to the Fourier method of separation of variables, we have to solve the following problem:

$$\begin{aligned} & Z_{ij}''(z) + \lambda_{ij}^2 Z_{ij}(z) = a_{ij}(z); \\ & \lambda_{ij}^2 = \lambda_{xi}^2 + \lambda_{yj}^2 + k^2, \quad k^2 = 3\varepsilon^2(\Lambda - 1)\left(\Lambda\frac{g}{3} - 1\right); \\ & Z_{ij}'(z) - \eta Z_{ij}(z) = 0, \quad z = 0; \\ & Z_{ij}'(z) + \eta Z_{ij}(z) = 0, \quad z = c. \end{aligned} \tag{32}$$

Here $a_{ij}(z)$ is the coefficient of the Fourier function $-K\exp(-\varepsilon z)$ expanded in terms of the orthonormalized system $\left\{\overline{X}_i(x)\overline{Y}_j(y)\right\}_{i,j=1}^{\infty}$:

$$a_{ij}(z) = -K\exp(-\varepsilon z)\int_0^a\int_0^b \overline{X}_i(x)\,\overline{Y}_j(y)\,dxdy = -K\exp(-\varepsilon z)\,\alpha_{ij},$$

$$\alpha_{ij} = \frac{1}{\|X_i\|}\int_0^a X_i(x)\,dx\,\frac{1}{\|Y_j\|}\int_0^b Y_j(y)\,dy.$$

Denote

$$n_{X_i} = \frac{1}{\|X_i\|}\int_0^a X_i(x)\,dx = \frac{2}{\lambda_{xi}}\sin\frac{a\lambda_{xi}}{2}\sin\left(\frac{a\lambda_{xi}}{2} + \arctan\left(\frac{\lambda_{xi}}{\eta}\right)\right),$$

$$n_{Y_j} = \frac{1}{\|Y_j\|}\int_0^b Y_j(y)\,dy = \frac{2}{\lambda_{yj}}\sin\frac{b\lambda_{yj}}{2}\sin\left(\frac{b\lambda_{yj}}{2} + \arctan\left(\frac{\lambda_{yj}}{\eta}\right)\right).$$

Then the right-hand side of Eq. (32) will have the form

$$a_{ij}(z) = -K\exp(-\varepsilon z)\frac{n_{X_i} n_{Y_j}}{\sqrt{\left(\frac{a}{2} + \frac{\eta}{\lambda_{xi}^2 + \eta^2}\right)\left(\frac{b}{2} + \frac{\eta}{\lambda_{yj}^2 + \eta^2}\right)}}. \tag{33}$$

If $\lambda_{ij} \neq \varepsilon$, then we solve (32) as

$$Z_{ij}(z) = C_1 \exp(-\lambda_{ij} z) + C_2 \exp(\lambda_{ij} z) + \beta \exp(-\lambda_{ij} z), \quad \beta = \frac{K\alpha_{ij}}{\lambda_{ij}^2 - \varepsilon^2}, \tag{34}$$

where

$$C_1 = \frac{\beta\left(\varepsilon + \eta - \dfrac{(\lambda_{ij} - \eta)(\varepsilon - \eta)}{(\lambda_{ij} + \eta)} \exp(-c(\lambda_{ij} + \varepsilon))\right)}{(\lambda_{ij} + \eta)\left[-1 + \dfrac{(\lambda_{ij} - \eta)^2}{(\lambda_{ij} + \eta)^2} \exp(-2\lambda_{ij} c)\right]}; \tag{35}$$

$$C_2 = \frac{\beta \exp(-\lambda_{ij} c)\left((\exp(-\lambda_{ij} c) - \exp(-\varepsilon c))(\lambda_{ij}\varepsilon - \eta^2) + (\exp(-\lambda_{ij} c) + \exp(-\varepsilon c))(\lambda_{ij}\eta - \eta\varepsilon)\right)}{(\lambda_{ij} + \eta)^2\left[-1 + \dfrac{(\lambda_{ij} - \eta)^2}{(\lambda_{ij} + \eta)^2} \exp(-2\lambda_{ij} c)\right]}. \tag{36}$$

In so doing, the solution of the boundary-value problem (24) for the modified Helmholz equation will be as follows:

$$J_0(x, y, z) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} Z_{ij}(z) \frac{X_i(x)\, Y_j(y)}{\sqrt{\left(\dfrac{a}{2} + \dfrac{\eta}{\lambda_{xi}^2 + \eta^2}\right)\left(\dfrac{b}{2} + \dfrac{\eta}{\lambda_{yj}^2 + \eta^2}\right)}}. \tag{37}$$

The functions $J_1(\mathbf{r})$, $J_2(\mathbf{r})$, and $J_3(\mathbf{r})$ are determined by formulas (14), (15), and (16) upon substitution of (37) for $J_0(\mathbf{r})$. Thus, the solution of the radiation-transfer equation will be of the form

$$I(x, y, z, \Omega_1, \Omega_2, \Omega_3) = J_0(x, y, z) + J_1(x, y, z)\,\Omega_1 + J_2(x, y, z)\,\Omega_2 + J_3(x, y, z)\,\Omega_3. \tag{38}$$

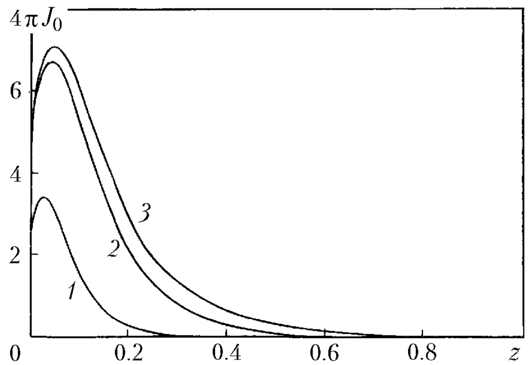
For illustration, we give examples of using the obtained solution (38) for calculating the light fluxes formed in a limited volume of a dispersion medium.

Example 1. Consider the case where the dispersion medium has the form of a rectangular parallelepiped with dimensions $a \times b \times c$. Choose the following parameters of the problem: $g = 0.99$, the volume attenuation index $\varepsilon = 25$, the single scattering albedo $\Lambda = 0.95$, and $F = 1$ (the luminous flux incident on the dispersion medium volume will be equal to π). The parallelepiped height will be assumed to be $c = 1$ and its length and width will be varied. The dimensionality a, b, c is chosen for convenience so that the products εa, εb, εc are dimensionless quantities.

Integrating the RTE solution (38) with respect to the solid angle, we obtain the scalar flux $4\pi J_0(x, y, z)$ inside the dispersion medium. The dependence of the scalar flux on the variable z at fixed values of $x = a/2$ and $y = b/2$ is shown in Fig. 1 for a = b.

In calculating the values of the function $J_0(x, y, z)$, we used 400 summands in sum (37). The dependence has a characteristic form with a maximum, whose value increases to some fixed value with increasing cross section of the medium.

Example 2. The dependence of $4\pi J_0(x, y, z)$ on the variable z at fixed values of $x = a/2$ and $y = b/2$ for the $2 \times 2 \times 1$ rectangular parallelepiped at various values of the attenuation index ε is given in Fig. 2. It is seen that with increasing attenuation index the optical radiation is localized closer to the lower boundary of the medium $z = 0$.



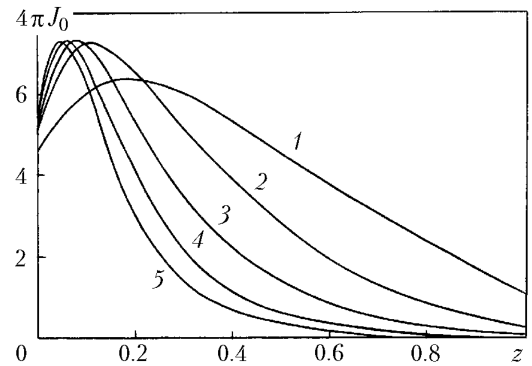
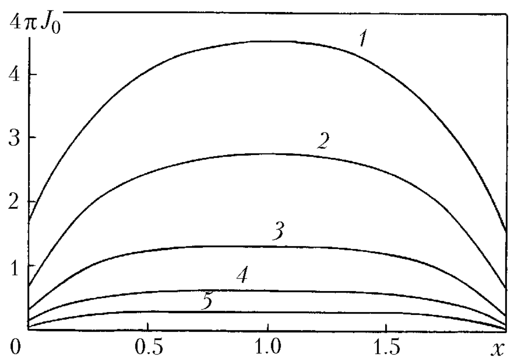


Fig. 1. Function $4\pi J_0(a/2,\ b/2,\ z)$ versus z: 1) $a = 0.125$; 2) 0.5; 3) 4.0.

Fig. 2. Function $4\pi J_0(a/2,\ b/2,\ z)$ versus z: 1) $\varepsilon = 5$; 2) 10; 3) 15; 4) 20; 5) 25.



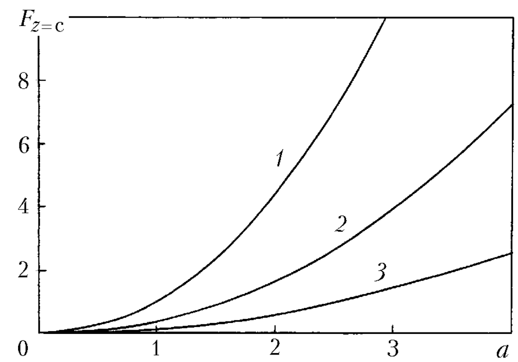


Fig. 3. Function $4\pi J_0(x,\ b/2,\ c/2)$ versus x: 1) $\varepsilon = 5$; 2) 10; 3) 15; 4) 20; 5) 25.

Fig. 4. Luminous flux $F_{z=c}$ emergent from the face $z = c$ versus the parallelepiped length: 1) $c = 0.125$; 2) 0.25; 3) 0.375.

Example 3. The dependence of the scalar flux $4\pi J_0(x,\ y,\ z)$ on the x-coordinate at fixed values of $z = c/2$ and $y = b/2$ for the $2 \times 2 \times 1$ rectangular parallelepiped at various values of the attenuation index ε is given in Fig. 3. Here a growth of the luminous flux gradients at the lateral boundaries of the medium on going from high to low values of the attenuation index is observed.

Example 4. Figure 4 shows the luminous flux $F_{z=c}$ emergent from the face $z = c$ versus the geometrical dimensions of the medium on the assumption of $a = b$ at various values of c. Importantly, the luminous flux emergent from the upper face $z = c$ is not only determined by ε and c, but also strongly depends on the cross section of the medium.

Using solution (38) of the boundary-value problem for the RTE, it is easy to calculate the directional luminous radiation fluxes emergent from each face of the parallelepiped:
from the right lateral face ($x = a$):

$$F_{x=a} = \int_0^c \int_0^b \left\{ \pi J_0 (a, y, z) + \frac{2}{3} \pi J_1 (a, y, z) \right\} dydz ;$$

from the left face ($x = 0$) $F_{x=0} = F_{x=a}$;
from the lateral face ($y = a$):

$$F_{y=b} = \int_0^c \int_0^a \left\{ \pi J_0 (x, b, z) + \frac{2}{3} \pi J_2 (x, b, z) \right\} dxdz ;$$

TABLE 1. Fluxes (%) Emergent from Each Face of the Parallelepiped

Luminous flux	Geometrical dimensions of the dispersion medium		
	$1 \times 1 \times 3$	$3 \times 1 \times 1$	$1 \times 1 \times 1$
$F_{x=0}$	6.25	1.81	6.25
$F_{x=a}$	6.25	1.81	6.25
$F_{y=0}$	6.25	6.68	6.25
$F_{y=b}$	6.25	6.68	6.25
$F_{z=0}$	48.19	51.16	48.19
$F_{z=c}$	0	0.02	0.01
F_s	73.18	68.16	73.19
F_{abs}	26.82	31.84	26.81
F_i	100	100	100

from the face $y = 0$ $F_{y=0} = F_{y=b}$;
from the upper face ($z = c$):

$$F_{z=c} = \int_0^a \int_0^b \left\{ \pi J_0 (x, y, c) + \frac{2}{3} \pi J_3 (x, y, c) \right\} dxdy + \pi Fab \exp(-\varepsilon c) ;$$

from the lower face ($z = 0$):

$$F_{z=0} = \int_0^a \int_0^b \left\{ \pi J_0 (x, y, 0) - \frac{2}{3} \pi J_3 (x, y, 0) \right\} dxdy ,$$

as well as the luminous flux from the source incident on the lower face $F_i = \pi Fab$.

Apparently, the total scattered luminous flux $F_s = F_{x=0} + F_{x=a} + F_{y=0} + F_{y=b} + F_{z=0} + F_{z=0} + F_{z=c}$, and the luminous flux absorbed by the medium $F_{abs} = F_i - F_s$.

The results of the numerical experiment ($\varepsilon = 25$, $\Lambda = 0.95$, $F = 1$, $g = 0.99$) on determination of fluxes (in %) emergent from each face for parallelepipeds of various dimensions are presented in Table 1.

Conclusions. A solution of the equation of radiation transfer in a dispersion medium of limited volume with a linear scattering indicatrix has been obtained. To this end, we used the transition to the boundary-value problem for the modified Helmholz equation, which was solved by the Fourier method with the use of a MatLab mathematical package. The RTE solution is given at the sum of four terms. The numerical calculations have shown that, apart from the medium characteristics, the value of the scalar and directional luminous fluxes is strongly influenced by the medium boundaries.

The results obtained for the scattered radiation characteristics can be used in solving various applied problems connected with multiple scattering of optical radiation beams in natural dispersion objects, including the cases where the medium volume is strongly limited and where it is necessary to take into account the influence of boundaries on the structure of the scattered radiation.

NOTATION

a and b, length and width of the parallelepiped; $B_1(\mathbf{r}, \mathbf{\Omega})$, function of sources; c, height of the parallelepiped; F_i, incident luminous flux; F_s and F_{abs}, scattered and absorbed luminous fluxes; $F_{x=0}$, $F_{x=a}$, $F_{y=0}$, $F_{y=b}$, $F_{z=0}$, and $F_{z=c}$, luminous fluxes emergent from faces $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, respectively; $J_0(\mathbf{r})$, mean spherical intensity of diffuse radiation; $J_1(\mathbf{r})$, $J_2(\mathbf{r})$, and $J_3(\mathbf{r})$, directional luminous fluxes along the axes x, y, and z;

$\mathbf{n}_\Gamma$, vector of the normal to the medium boundary; $4\pi J_0(\mathbf{r})$, scalar radiation flux; Γ, medium boundary; ε, volume attenuation index; $\kappa(\mathbf{\Omega}, \mathbf{\Omega}')$, light-scattering indicatrix; Λ, albedo of single scattering; $\mathbf{\Omega}$, unit vector of the beam direction; Ω_1, Ω_2, and Ω_3, direction cosines of vector $\mathbf{\Omega}$ with respect to the *x*-, *y*-, and *z*-coordinates. Subscripts: abs, absorbed; i, incident; s, scattered.

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